PRACTICAL NUMBERS IN LUCAS SEQUENCES

CARLO SANNA

ABSTRACT. A *practical number* is a positive integer n such that all the positive integers $m \leq n$ can be written as a sum of distinct divisors of n. Let $(u_n)_{n \geq 0}$ be the Lucas sequence satisfying $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = au_{n+1} + bu_n$ for all integers $n \geq 0$, where a and b are fixed nonzero integers. Assume $a(b+1)$ even and $a^2 + 4b > 0$. Also, let $\mathcal{A}$ be the set of all positive integers n such that $|u_n|$ is a practical number. Melfi proved that $\mathcal{A}$ is infinite. We improve this result by showing that $\#\mathcal{A}(x) \gg x/\log x$ for all $x \geq 2$, where the implied constant depends on a and b. We also pose some open questions regarding $\mathcal{A}$.

1. Introduction

A *practical number* is a positive integer n such that all the positive integers $m \leq n$ can be written as a sum of distinct divisors of n. The term "practical" was coined by Srinivasan [7]. Let $\mathcal{P}$ be the set of practical numbers. Estimates for the counting function $\#\mathcal{P}(x)$ were given by Hausman and Shapiro [1], Tenenbaum [10], Margenstern [2], Saias [5], and, finally, Weingartner [12], who proved that there exists a constant $C > 0$ such that

$$\#\mathcal{P}(x) = \frac{Cx}{\log x} \cdot \left(1 + O\left(\frac{\log\log x}{\log x}\right)\right)$$

for all $x \geq 3$, settling a conjecture of Margenstern [2].

In analogy with well-known conjectures about prime numbers, Melfi [4] proved that every positive even integer is the sum of two practical numbers, and that there are infinitely many triples $(n, n+2, n+4)$ of practical numbers. Let $(u_n)_{n \geq 0}$ be a Lucas sequence, that is, a sequence of integers satisfying $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = au_{n+1} + bu_n$ for all integers $n \geq 0$, where a and b are two fixed nonzero integers. Also, let $\mathcal{A}$ be the set of all positive integers n such that $|u_n|$ is a practical number. From now on, we assume $a^2 + 4b > 0$ and $a(b+1)$ even. We remark that, in the study of $\mathcal{A}$, assuming $a(b+1)$ even is not a loss of generality. Indeed, if $a(b+1)$ is odd then u_n is odd for all $n \geq 1$ and, since 1 is the only odd practical number, it follows that $\mathcal{A} = \{1\}$. Melfi [3, Theorem 10] proved the following result.

Theorem 1.1. *The set $\mathcal{A}$ is infinite. Precisely, $2^j \cdot 3 \in \mathcal{A}$ for all sufficiently large positive integers j, how large depending on a and b, and hence*

$$\#\mathcal{A}(x) \gg \log x,$$

for all sufficiently large $x > 1$.

In this paper, we improve Theorem 1.1 to the following:

Theorem 1.2. *For all $x \geq 2$, we have*

$$\#\mathcal{A}(x) \gg \frac{x}{\log x},$$

where the implied constant depends on a and b.

We leave the following open questions to the interested readers:

(Q1) Does $\mathcal{A}$ have zero natural density?

(Q2) Can a nontrivial upper bound for $\#\mathcal{A}(x)$ be proved?

2010 *Mathematics Subject Classification.* Primary: 11B37, Secondary: 11B39, 11N25.

Key words and phrases. Fibonacci numbers; Lucas sequences; practical numbers.

(Q3) Are there infinitely many nonpractical n such that $|u_n|$ is practical?

(Q4) Are there infinitely many practical n such that $|u_n|$ is nonpractical?

(Q5) What about practical numbers in general integral linear recurrences over the integers?

Notation. For any set of positive integers $\mathcal{S}$, we put $\mathcal{S}(x) := \mathcal{S} \cap [1, x]$ for all $x \geq 1$, and $\#\mathcal{S}(x)$ denotes the counting function of $\mathcal{S}$. We employ the Landau–Bachmann "Big Oh" notation O, as well as the associated Vinogradov symbols $\ll$ and $\gg$, with their usual meanings. Any dependence of the implied constants is explicitly stated. As usual, we write $\mu(n)$, $\varphi(n)$, $\sigma(n)$, and $\omega(n)$, for the Möbius function, the Euler's totient function, the sum of divisors, and the number of prime factors of a positive integer n, respectively.

2. Preliminaries on Lucas sequences

In this section we collect some basic facts about Lucas sequences. Let α and β be the two roots of the characteristic polynomial $X^2 - aX - b$. Since $a^2 + 4b > 0$ and $b \neq 0$, we have that α and β are real, nonzero, and distinct. It is well known that the *generalized Binet's formula*

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1}$$

holds for all integers $n \geq 0$. Define

$$\Phi_n := \prod_{\substack{1 \leq k \leq n \\ \gcd(n,k) = 1}} \left(\alpha - e^{2\pi \mathbf{i} k/n} \beta\right),$$

for each positive integer n. It can be proved that $\Phi_n \in \mathbb{Z}$ for all integers $n > 1$ (see, e.g., [9, p. 428]). Furthermore, we have

$$u_n = \prod_{\substack{d \,|\, n \\ d > 1}} \Phi_d \tag{2}$$

and, by the Möbius inversion formula,

$$\Phi_n = \prod_{d \,|\, n} u_{n/d}^{\mu(d)} \tag{3}$$

for all integers $n > 1$. Changing the sign of a changes the signs of α, β and turns u_n into $(-1)^{n+1} u_n$, which is not a problem, since for the study of $\mathcal{A}$ we are interested only in $|u_n|$. Hence, without loss of generality, we can assume $a > 0$ and $\alpha > |\beta|$, which in turn implies that $u_n, \Phi_n > 0$ for all integers $n > 0$. We conclude this section with an easy lemma regarding the growth of u_n and Φ_n.

Lemma 2.1. *For all integers $n > 0$, we have*

(i) $u_n \geq u_{n-1}$*;*

(ii) $u_n = \alpha^{n + O(1)}$*;*

(iii) $\Phi_n = \alpha^{\varphi(n) + O(1)}$*;*

where the implied constants depend on a and b.

Proof. If $b > 0$, then (i) is clear from the recursion for u_n. Hence, suppose $b < 0$, so that $\beta > 0$. After a bit of manipulations, (i) is equivalent to $\alpha^{n-1}(\alpha - 1) \geq \beta^{n-1}(\beta - 1)$, which in turn follows easily since $\alpha > \beta > 0$. Claim (ii) is a consequence of (1). Setting $\gamma := \beta/\alpha$, by (1) and (3), we get

$$\Phi_n = \alpha^{\varphi(n)} \prod_{d \,|\, n} \left(\frac{1 - \gamma^{n/d}}{\alpha - \beta}\right)^{\mu(d)} = \alpha^{\varphi(n)} \prod_{d \,|\, n} \left(1 - \gamma^{n/d}\right)^{\mu(d)},$$

for all integers $n > 1$, where we used the well-known formulas $\sum_{d \mid n} \mu(d) \frac{n}{d} = \varphi(n)$ and $\sum_{d \mid n} \mu(d) = 0$. Therefore, since $|\gamma| < 1$, we have

$$|\log(\Phi_n/\alpha^{\varphi(n)})| \leq \sum_{d \mid n} |\log(1-\gamma^d)| \ll \sum_{d=1}^{\infty} |\gamma|^d \ll 1,$$

and also (iii) is proved. □

3. Preliminaries on practical numbers and close relatives

The following lemma on practical numbers will be fundamental later.

Lemma 3.1. *If n is a practical number and $m \leq 2n$ is a positive integer, then mn is a practical number.*

Proof. See [4, Lemma 1]. □

Close relatives of practical numbers are *φ-practical numbers*. A φ-practical number is a positive integer n such that all the positive integers $m \leq n$ can be written as $m = \sum_{d \in \mathcal{D}} \varphi(d)$, where $\mathcal{D}$ is a subset of the divisors of n. This notion was introduced by Thompson [11] while studying the degrees of the divisors of the polynomial $X^n - 1$. Indeed, φ-practical numbers are exactly the positive integers n such that $X^n - 1$ has a divisor of every degree up to n.

We need a couple of results regarding φ-practical numbers.

Lemma 3.2. *Let n be a φ-practical number and p be a prime number not dividing n. Then pn is φ-practical if and only if $p \leq n+2$. Moreover, $p^j n$ is φ-practical if and only if $p \leq n+1$, for every integer $j \geq 2$.*

Proof. See [11, Lemma 4.1]. □

Lemma 3.3. *If n is an even φ-practical number, and if $d_1, \dots, d_s$ are all the divisors of n ordered so that $\varphi(d_1) \leq \dots \leq \varphi(d_s)$, then*

$$\varphi(d_{j+1}) \leq \sum_{i=1}^{j} \varphi(d_i), \tag{4}$$

for all positive integers $j < s$.

Proof. It is not difficult to see that n is φ-practical if and only if

$$\varphi(d_{j+1}) \leq 1 + \sum_{i=1}^{j} \varphi(d_i), \tag{5}$$

for all positive integers $j < s$ (see [11, p. 1041]). Hence, we have only to prove that n even ensures that in (5) the equality cannot happen. If $j = 1$ then (4) is obvious since $\{d_1, d_2\} = \{1, 2\}$, so we can assume $1 < j < s$. At this point $\varphi(d_{j+1})$ is even, while

$$1 + \sum_{i=1}^{j} \varphi(d_i)$$

is odd, because $\varphi(m)$ is even for all integers $m > 2$. Thus, in (5) the equality is not satisfied. □

Let θ be a real-valued arithmetic function, and define $\mathcal{B}_\theta$ as the set containing $n = 1$ and all those $n = p_1^{a_1} \cdots p_k^{a_k}$, where $p_1 < \dots < p_k$ are prime numbers and $a_1, \dots, a_k$ are positive integers, which satisfy

$$p_j \leq \theta\left(\prod_{i=1}^{j-1} p_i^{a_i}\right),$$

for $j = 1, \dots, k$, where the empty product is equal to 1. If $\theta(n) := \sigma(n) + 1$, then $\mathcal{B}_\theta$ is the set of practical numbers. This is a characterization given by Stewart [8] and Sierpiński [6].

Weingartner proved a general and strong estimate for $\#\mathcal{B}_\theta(x)$. The following is a simplified version adapted just for our purposes.

Theorem 3.4. *Suppose $\theta(1) \geq 2$ and $n \leq \theta(n) \leq An$ for all positive integers n, where $A \geq 1$ is a constant. Then, we have*

$$\#\mathcal{B}_\theta(x) \sim \frac{c_\theta x}{\log x},$$

as $x \to +\infty$, where $c_\theta > 0$ is a constant.

Proof. See [12, Theorems 1.2 and 5.1]. □

4. Proof of Theorem 1.2

The key tool of the proof is the following technical lemma.

Lemma 4.1. *Suppose that n is a sufficiently large positive integer, how large depending on a and b. Let p be a prime number and write $n = p^v m$ for some nonnegative integer v and some positive integer m not divisible by p. If m is an even φ-practical number, $n \in \mathcal{A}$, and $p < m$, then $p^k n \in \mathcal{A}$ for all positive integers k.*

Proof. Clearly, it is enough to prove the claim for $k = 1$. Let $d_1 = 1, d_2 = 2, \dots, d_s$ be all the divisors of m, ordered to that $\varphi(d_1) \leq \cdots \leq \varphi(d_s)$. Furthermore, define

$$N_j := u_n \prod_{i=1}^{j} \Phi_{p^{v+1} d_i},$$

for $j = 1, \dots, s$. We shall prove that each N_j is practical. This implies the thesis, since $N_s = u_{pn}$ by (2).

We proceed by induction on j. First, by (2) and Lemma 2.1(i), we have

$$\Phi_{p^{v+1} d_1} = \Phi_{p^{v+1}} \leq u_{p^{v+1}} \leq u_{p^v m} = u_n,$$

since $p < m$. Hence, applying Lemma 3.1 and the fact that u_n is practical, we get that $N_1 = u_n \Phi_{p^{v+1} d_1}$ is practical.

Now assuming that N_j is practical we shall prove that N_{j+1} is practical. Again, since $N_{j+1} = \Phi_{p^{v+1} d_{j+1}} N_j$, thanks to Lemma 3.1 it is enough to show that the inequality

$$\Phi_{p^{v+1} d_{j+1}} \leq u_n \prod_{i=1}^{j} \Phi_{p^{v+1} d_i} \tag{6}$$

holds. In turn, by Lemma 2.1(ii) and (iii), we have that (6) is implied by

$$n + \varphi(p^{v+1}) \left[-\varphi(d_{j+1}) + \sum_{i=1}^{j} \varphi(d_i) \right] \geq C(j+1), \tag{7}$$

where $C > 0$ is a constant depending only on a and b.

On the one hand, since m is an even φ-practical number, by Lemma 3.3 we have that the term of (7) in square brackets is nonnegative. On the other hand, for sufficiently large n, we have

$$n \geq C(\log n / \log 2 + 1) \geq C(\omega(n) + 1) \geq C(j+1).$$

Therefore, (7) holds and the proof is complete. □

We are ready to prove Theorem 1.2. Pick a sufficiently large positive integer h, depending on a and b, such that the claim of Lemma 4.1 holds for all integers $n \geq 2^h \cdot 3$. Moreover, by Theorem 1.1, we can assume that $2^j \cdot 3 \in \mathcal{A}$ for all integers $j \geq h$. Put $\mathcal{B} := \mathcal{B}_\theta \setminus \{1\}$, where $\theta(n) := \max\{2, n\}$. Note that, as a consequence of Lemma 3.2, all the elements of $\mathcal{B}$ are even φ-practical numbers. We shall prove that for all $n \in \mathcal{B}$ we have $2^h \cdot 3n \in \mathcal{A}$. In this way, thanks to Theorem 3.4, we get

$$\#\mathcal{A}(x) \geq \#\mathcal{B}\Big(\frac{x}{2^h \cdot 3}\Big) \gg \frac{x}{\log x},$$

for all sufficiently large x. Hence, since $1 \in \mathcal{A}$, Theorem 1.2 follows.

We proceed by induction on the number of prime factors of $n \in \mathcal{B}$. If $n \in \mathcal{B}$ has exactly one prime factor, then it follows easily that $n = 2^j$ for some positive integer j. Hence, we have $2^h \cdot 3n = 2^{h+j} \cdot 3 \in \mathcal{A}$, as claimed.

Now, assuming that the claim is true for all $n \in \mathcal{B}$ with exactly $k \geq 1$ prime factors, we shall prove it for all $n \in \mathcal{B}$ having $k+1$ prime factors. Write $n = p_1^{a_1} \cdots p_{k+1}^{a_{k+1}}$, where $p_1 < \cdots < p_{k+1}$ are prime numbers and $a_1, \dots, a_{k+1}$ are positive integers. Put also $m := p_1^{a_1} \cdots p_k^{a_k}$. Since $n \in \mathcal{B}$, we have $m \in \mathcal{B}$ and $p_{k+1} < m$. On the one hand, by the induction hypothesis, $2^h \cdot 3m \in \mathcal{A}$. On the other hand, it is easy to see that $m \in \mathcal{B}$ implies $2^h m \in \mathcal{B}$ and $2^h \cdot 3m \in \mathcal{B}$.

First, suppose $p_{k+1} > 3$. Since $2^h \cdot 3m$ is an even φ-practical number, $2^h \cdot 3m \in \mathcal{A}$, and $p_{k+1} < 2^h \cdot 3m$, by Lemma 4.1 we get that $2^h \cdot 3n = 2^h \cdot 3mp_{k+1}^{a_{k+1}} \in \mathcal{A}$, as claimed.

On the other hand, if $p_{k+1} = 3$ the situation is similar. Since $2^h m$ is an even φ-practical number, $2^h \cdot 3m \in \mathcal{A}$, and $p_{k+1} < 2^h m$, by Lemma 4.1 we get that $2^h \cdot 3n = 2^h \cdot 3mp_{k+1}^{a_{k+1}} \in \mathcal{A}$, as claimed. The proof is complete.

Acknowledgements. The authors thanks the anonymous referee for carefully reading the paper. The author is a member of the INdAM group GNSAGA.

References

1. M. Hausman and H. N. Shapiro, *On practical numbers*, Comm. Pure Appl. Math. **37** (1984), no. 5, 705–713.
2. M. Margenstern, *Les nombres pratiques: théorie, observations et conjectures*, J. Number Theory **37** (1991), no. 1, 1–36.
3. G. Melfi, *A survey on practical numbers*, Rend. Sem. Mat. Univ. Politec. Torino **53** (1995), no. 4, 347–359, Number theory, II (Rome, 1995).
4. G. Melfi, *On two conjectures about practical numbers*, J. Number Theory **56** (1996), no. 1, 205–210.
5. E. Saias, *Entiers à diviseurs denses. I*, J. Number Theory **62** (1997), no. 1, 163–191.
6. W. Sierpiński, *Sur une propriété des nombres naturels*, Ann. Mat. Pura Appl. (4) **39** (1955), 69–74.
7. A. K. Srinivasan, *Practical numbers*, Current Sci. **17** (1948), 179–180. MR 0027799
8. B. M. Stewart, *Sums of distinct divisors*, Amer. J. Math. **76** (1954), 779–785.
9. C. L. Stewart, *On divisors of Fermat, Fibonacci, Lucas, and Lehmer numbers*, Proc. London Math. Soc. (3) **35** (1977), no. 3, 425–447.
10. G. Tenenbaum, *Sur un problème de crible et ses applications*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 1, 1–30.
11. L. Thompson, *Polynomials with divisors of every degree*, J. Number Theory **132** (2012), no. 5, 1038–1053.
12. A. Weingartner, *Practical numbers and the distribution of divisors*, Q. J. Math. **66** (2015), no. 2, 743–758.

Università degli Studi di Torino, Department of Mathematics, Turin, Italy
E-mail address: carlo.sanna.dev@gmail.com